Visuospatial representation and arithmetical thinking in students with Special Educational Needs

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AIMS

Investigate the difficulties of low-attaining secondary-age students with multiplicative thinking, via their representational strategies for arithmetical problems.

Explore the use of nonstandard visuospatial representations in tuition, for improving struggling students’ understanding of multiplication and division.

Develop methods for qualitative diagnosis and assessment of arithmetical-representational strengths, weaknesses and microprogressions in students’ work.
Characteristics of research participants:

• aged 11-16
• in mainstream schools, placed in ‘bottom sets’
• particularly low-attaining and/or with SEND
• understood addition/subtraction, and could carry out some tasks with additive structure
• had some experience with multiplication/division – and negative feelings towards them
• could recall some multiplication facts
Characteristics of tasks used:

- natural numbers (max 3 digits)
- partitive and quotitive division
- multiplication (process and structure)
- scenario (e.g. biscuits, buses, clothing combinations)
- bare (e.g. number symbols, 2D and 3D arrays)
- psychologically commutative and non-commutative
The representational media available to participants were:

- paper and coloured felt tip pens
- coloured multilink cubes
- small bowls
- cut lengths of drinking straws
- own hands
- (internal)
SAMPLE REPRESENTATIONS: INITIAL

IMAGES NOT TO BE REPRODUCED
A person has 6 t-shirts and 4 pairs of trousers in their suitcase. The t-shirts are: white, blue, green, brown, red and yellow. The trousers are: black, blue, green and brown.

How many different possible outfits can be made?

(Example outfit: blue trousers with white t-shirt)

- What is the answer?
- How did you work it out?
- Why might a student get stuck?
- What might help them?
HOLIDAY CLOTHES – WRITTEN RESPONSES

Jenny, Y7

6 t-shirts - white, blue, green, brown, red, yellow
4 trousers - black, blue, green, brown

Danny, Y8

Tasha, Y8

all images from Finesilver (2009/2014)
HOLIDAY CLOTHES – DRAWN RESPONSES

Images not to be reproduced

Oscar, Y9
(5 mins)

Kieran, Y7

George, Y8

6 t-shirts:
white, blue, green, brown, red, yellow

4 trousers:
black, blue, green, brown

all images from Finesilver (2009/2014)
HOLIDAY CLOTHES – MIXED-MODE RESPONSES

IMAGES NOT TO BE REPRODUCED

Leo, Y7

all images from Finesilver (2009/2014)
HOLIDAY CLOTHES – INTERPRETATION

**Mode** e.g. modelling, drawing, writing, symbols, gesture

**Media** preferred materials, e.g. cubes, pen/paper, fingers

**Motion** e.g. static once created, or involving ongoing movement of elements

**Resemblance** between the drawing or model and the task scenario described (NB This is related to the more commonly used ‘abstraction’.)

**Consistency** i.e. whether a single representational strategy was used from start to finish, or changes occurred

**Completeness** i.e. whether the external representation had to be ‘finished’ for solution
“[C]hildren's knowledge of mathematics is extraordinarily complex and often much different from what we had supposed it to be . . . In the case of every child we have interviewed or observed, there have emerged startling contradictions, unsuspected strengths or weaknesses, and fascinating complexities.”

DIFFERENCES IN MATHEMATICAL ABILITIES

- Mathematical ‘ability’ is made up of many different components and subcomponents.
- There can be significant variation in how difficult a single individual, or individuals supposedly at the same ‘level’ find different subcomponents.
- Differences are due to specific patterns of strength and weakness, which may relate to SEN/D characteristics.
- Levels of actual performance on any occasion are influenced by factors which may be:
  - Internal to the individual (cognitive, affective)
  - External (environmental, pedagogical)
“[Word problems are] the castor oil of the mathematics curriculum: fairly unpleasant but possibly good for you”

(Askew, 2003)

Is this fair?

It depends on various factors:

- the learner(s) – abilities, attitudes, experiences
- language used
- representational modes & media available
- imaginability of scenario
 TASK: CUBOID

How many of the unit cubes make up the block?

- What is the answer?
- How did you work it out?
- Why might a student get stuck?
- What might help them?
Leo (Y7) – first attempt

red = 4
blue = 2
green = 11
black = 5

Tasha (Y8) three different orientations

Leo (Y7) – second attempt ‘pulling out the drawers’

Paula (Y10) 3-colour block, 3-colour sum

all data from Finesilver (2014/2017)
## CUBOID – INTERPRETATION

<table>
<thead>
<tr>
<th>Enumeration: Do they work out the total number of cubes by...?</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Multiplication</td>
</tr>
<tr>
<td>• Addition/Step-counting</td>
</tr>
<tr>
<td>• Counting</td>
</tr>
<tr>
<td>• Step counting</td>
</tr>
<tr>
<td>• Rhythmic counting</td>
</tr>
<tr>
<td>• Grouped counting</td>
</tr>
<tr>
<td>• Unitary counting</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spatial structuring: Do they conceptualise the cubes as...?</th>
</tr>
</thead>
<tbody>
<tr>
<td>• 3D multiplicative structure</td>
</tr>
<tr>
<td>• stacked set of 2D layers</td>
</tr>
<tr>
<td>• arrayed set of 1D columns</td>
</tr>
<tr>
<td>• set of faces (surface area)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Errors: If they go wrong, is it...?</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Spatial structuring</td>
</tr>
<tr>
<td>• Numeric calculation</td>
</tr>
<tr>
<td>• Fact retrieval</td>
</tr>
<tr>
<td>• Verbal count sequence</td>
</tr>
<tr>
<td>• Visuospatial/kinaesthetic</td>
</tr>
</tbody>
</table>
Some of the major skills called upon:

- Comparison and estimation of quantities
- Interpretation and manipulation of arbitrary symbols
- Memory (verbal & visuospatial)
- Sequencing
- Pattern recognition
- Task-based strategizing
- ...

Learners with SEN/D need to be metastrategic – not rely on skills which are (currently) weak, and make use of strengths.
POTENTIAL BARRIERS

• No ‘clean slates’ – standard teaching approaches have been tried (maybe many times)

• Fear and loathing attached to operations, from past failure – avoidance

• Shame in admitting to lack of understanding/capability – concealment of ‘immature’ strategies

• Belief that there is only a single acceptable method for a given task type (from teachers and peer pressure)

• *Sunk cost bias* – not wanting previous hard work to ‘go to waste’, so persisting with unsuccessful strategies
SAMPLE REPRESENTATIONS: 'BISCUITS'

IMAGES NOT TO BE REPRODUCED
SAMPLE REPRESENTATIONS: ‘PASSENGERS’

IMAGES NOT TO BE REPRODUCED
# Basic Qualitative Analytical Framework

## Types of representations created:
- **Media** e.g. cubes, pen/paper
- **Mode** e.g. modelling, drawing, words, symbols
- **Resemblance** between the drawing or model and the task scenario described

## Relationship between representation and calculation:
- **Motion** e.g. static once created, or involving ongoing movement of elements
- **Unitariness** i.e. whether one representational unit (e.g. cube, tally mark) represents exactly one scenario/numerical unit, or may stand for a group
- **Spatial structure** e.g. grouping of units through separation in space, use of containers, aligning in one or more dimensions
- **Consistency** i.e. whether a single coherent strategy was used from start to finish, or changes occurred
- **Completeness** i.e. whether the external representation had to be ‘finished’ for solution
- **Enumeration** e.g. unit-counting, step-counting, number fact retrieval
- **Errors** e.g. incorrectly-retrieved number fact, verbal count error
- **Success** i.e. whether the strategy produced a correct solution

## Teacher-student interaction:
- **Verbal** e.g. spoken prompts, suggestions for calculation
- **Visuospatial** e.g. gestures, participation in modelling/drawing
**Unit container:** Groups of two or more units enclosed by visible boundaries. Includes representations where units are aligned in rows and/or columns, but these do not represent divisor/quotient or multiplier/multiplicand.
Unit containers (2)
Unit array: Groups of two or more units aligned in rows and columns, where number of units in the rows/columns represents divisor/quotient or multiplier/multiplicand.
Array-container blend: Unit array representation with additional containing rings, where number of units in each row/column/container represents divisor/quotient or multiplier/multiplicand.
Array-container blends (2)

$4 \times 7 = 28$
$9 \times 4 = 28$
$28 \div 4 = 7$
$28 \div 7 = 4$
**Number container:** Container representation with numerals (rather than unit marks) representing the number in each group written inside, or close by, each container.
Images not to be reproduced
<table>
<thead>
<tr>
<th>Representation type</th>
<th>Students’ current stage</th>
<th>Subject content</th>
<th>How it helped</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unit containers</strong></td>
<td>Partial or no concept of division</td>
<td>Partitive and quotitive division; relationship with multiplication</td>
<td>Model for sharing and grouping units; seeing repetitive structure</td>
</tr>
<tr>
<td><strong>Unit arrays</strong></td>
<td>Partial or procedural concept of division</td>
<td>As above, plus: commutativity of multiplication; multiplicative structures as static relationships</td>
<td>As above, but seeing 2-dimensional repetitive structure</td>
</tr>
<tr>
<td><strong>Array-container blends</strong></td>
<td>Using either unit containers or arrays to divide</td>
<td>As above, plus: factorising numbers</td>
<td>As above, plus: seeing equal groups as ‘units’ in a larger structure</td>
</tr>
<tr>
<td><strong>Number containers</strong></td>
<td>Ready to transition from unitary to symbolic representation</td>
<td>Quotitive division; multiplication; recording work</td>
<td>Introducing symbolic notation via familiar imagery</td>
</tr>
</tbody>
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